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Closed Forms of Two Types of Fractional Integrals

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we find the closed forms of two types of fractional integrals by using some methods. Moreover, our results are generalizations of classical calculus results.**

Keywords: **Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, closed forms, fractional integrals.**

I. INTRODUCTION

Fractional calculus is the theory of non-integer derivative and integral. However, the definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald Letnikov (G-L) fractional derivative and Jumarie's modification of R-L fractional derivative [1-5]. In the past decades, fractional calculus has been widely used in continuum mechanics, quantum mechanics, electronic engineering, fluid science, viscoelasticity, control theory, dynamics, financial economics and other fields [6-17].

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we use some techniques to find the closed forms of the following two types of fractional integrals:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[-\frac{1}{2}\,sin_{\alpha}(x^{\alpha})\otimes_{\alpha}Ln_{\alpha}(1+2rcos_{\alpha}(x^{\alpha})+r^{2})-cos_{\alpha}(x^{\alpha})\otimes_{\alpha}arctan_{\alpha}\left(rsin_{\alpha}(x^{\alpha})\otimes_{\alpha}[1+rcos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}(-1)}\right)\right],$$

and

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\frac{1}{2}\,cos_{\alpha}(x^{\alpha})\otimes_{\alpha}Ln_{\alpha}(1+2rcos_{\alpha}(x^{\alpha})+r^{2})-sin_{\alpha}(x^{\alpha})\otimes_{\alpha}arctan_{\alpha}\left(rsin_{\alpha}(x^{\alpha})\otimes_{\alpha}[1+rcos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}(-1)}\right)\right],$$

where $0<\alpha\leq 1$, and r is a real number. In fact, our results are generalizations of ordinary calculus results.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper and its properties.

Definition 2.1 ([18])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_{x}^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,, \quad (1)$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_{x}^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \quad (2)$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([19])**:** *If* α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C]=0. \tag{4}$$

Definition 2.3 ([20])**:** Let x, x_0 and a_k be real numbers for all k, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at $x=x_0$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([21])**:** If $0<\alpha\leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x=x_0$,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes_\alpha\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}\otimes_\alpha\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \\ &=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \end{aligned} \tag{8}$$

Definition 2.5 ([22])**:** Assume that $0<\alpha\leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{9}$$

And the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2k}, \tag{10}$$

and

$$sin_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2k+1)}. \tag{11}$$

Theorem 2.6 (fractional Euler's formula)([23])**:** *If* $0<\alpha\leq 1$, *and* $i=\sqrt{-1}$, *then*

$$E_\alpha(ix^\alpha)=cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha). \tag{12}$$

Notation 2.7: If the complex number $z=p+iq$, where p,q are real numbers. p is the real part of z, and denoted by $\text{Re}(z)$; q is the imaginary part of z, and denoted by $\text{Im}(z)$.

III. MAIN RESULTS

In this section, we use some methods to find the closed forms of two types of fractional integrals. At first, a lemma is needed.

Lemma 3.1: *If* $0<\alpha\leq 1$, *and* r *is a real number, then*

$$\text{Re}\left[\left(1+rE_\alpha(ix^\alpha)\right)\otimes_\alpha Ln_\alpha\left(1+rE_\alpha(ix^\alpha)\right)-rE_\alpha(ix^\alpha)\right]$$

$$= [1 + rcos_\alpha(x^\alpha)] \otimes_\alpha \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2)$$

$$-rsin_\alpha(x^\alpha) \otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right) - rcos_\alpha(x^\alpha) . \quad (13)$$

$$\text{Im}\left[\left(1 + rE_\alpha(ix^\alpha)\right) \otimes_\alpha Ln_\alpha\left(1 + rE_\alpha(ix^\alpha)\right) - rE_\alpha(ix^\alpha)\right]$$

$$= rsin_\alpha(x^\alpha) \otimes_\alpha \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2)$$

$$+[1 + rcos_\alpha(x^\alpha)] \otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right) - rsin_\alpha(x^\alpha) \quad (14)$$

Proof Since

$$\left(1 + rE_\alpha(ix^\alpha)\right) \otimes_\alpha Ln_\alpha\left(1 + rE_\alpha(ix^\alpha)\right) - rE_\alpha(ix^\alpha)$$

$$= \left(1 + rcos_\alpha(x^\alpha) + irsin_\alpha(x^\alpha)\right) \otimes_\alpha Ln_\alpha\left(1 + rcos_\alpha(x^\alpha) + irsin_\alpha(x^\alpha)\right) - rcos_\alpha(x^\alpha) - irsin_\alpha(x^\alpha)$$

$$= \left(1 + rcos_\alpha(x^\alpha) + irsin_\alpha(x^\alpha)\right) \otimes_\alpha$$

$$\left[\frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2) + i \cdot arctan_\alpha\left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)\right] - rcos_\alpha(x^\alpha) - irsin_\alpha(x^\alpha) . \quad (15)$$

Therefore,

$$\text{Re}\left[\left(1 + rE_\alpha(ix^\alpha)\right) \otimes_\alpha Ln_\alpha\left(1 + rE_\alpha(ix^\alpha)\right) - rE_\alpha(ix^\alpha)\right]$$

$$= [1 + rcos_\alpha(x^\alpha)] \otimes_\alpha \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2)$$

$$-rsin_\alpha(x^\alpha) \otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right) - rcos_\alpha(x^\alpha).$$

And

$$\text{Im}\left[\left(1 + rE_\alpha(ix^\alpha)\right) \otimes_\alpha Ln_\alpha\left(1 + rE_\alpha(ix^\alpha)\right) - rE_\alpha(ix^\alpha)\right]$$

$$= rsin_\alpha(x^\alpha) \otimes_\alpha \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2)$$

$$+[1 + rcos_\alpha(x^\alpha)] \otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right) - rsin_\alpha(x^\alpha).$$

q.e.d.

Theorem 3.2: *If* $0 < \alpha \leq 1$, *and* r *is a real number, then*

$$\left({}_0I_x^\alpha\right)\left[-rsin_\alpha(x^\alpha) \otimes_\alpha \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2) - rcos_\alpha(x^\alpha) \otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)\right]$$

$$= [1 + rcos_\alpha(x^\alpha)] \otimes_\alpha \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2)$$

$$-rsin_\alpha(x^\alpha) \otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right) - rcos_\alpha(x^\alpha). \quad (16)$$

and

$$\left({}_0I_x^\alpha\right)\left[rcos_\alpha(x^\alpha) \otimes_\alpha \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2) - rsin_\alpha(x^\alpha) \otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)\right]$$

$$= rsin_\alpha(x^\alpha) \otimes_\alpha \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2)$$

$$+[1 + rcos_\alpha(x^\alpha)] \otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right) - rsin_\alpha(x^\alpha). \quad (17)$$

Proof Since

$$\left({}_0I_x^\alpha\right)\left[Ln_\alpha\left(1 + rE_\alpha(ix^\alpha)\right) \otimes_\alpha \left({}_0D_x^\alpha\right)[rE_\alpha(ix^\alpha)]\right] = \left(1 + rE_\alpha(ix^\alpha)\right) \otimes_\alpha Ln_\alpha\left(1 + rE_\alpha(ix^\alpha)\right) - rE_\alpha(ix^\alpha). \quad (18)$$

It follows that

$$\left({}_0I_x^\alpha\right)\left[Ln_\alpha\left(1+rE_\alpha(ix^\alpha)\right)\otimes_\alpha riE_\alpha(ix^\alpha)\right]=\left(1+rE_\alpha(ix^\alpha)\right)\otimes_\alpha Ln_\alpha\left(1+rE_\alpha(ix^\alpha)\right)-rE_\alpha(ix^\alpha). \quad (19)$$

And hence,

$$\left({}_0I_x^\alpha\right)\left[\left[\frac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2)+i\cdot arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right)\right]\otimes_\alpha[-rsin_\alpha(x^\alpha)+ircos_\alpha(x^\alpha)]\right]$$
$$=\left(1+rE_\alpha(ix^\alpha)\right)\otimes_\alpha Ln_\alpha\left(1+rE_\alpha(ix^\alpha)\right)-rE_\alpha(ix^\alpha). \quad (20)$$

Therefore,

$$\left({}_0I_x^\alpha\right)\left[-r\,sin_\alpha(x^\alpha)\otimes_\alpha\frac{1}{2}Ln_\alpha(1+2rcos_\alpha(x^\alpha)+r^2)-rcos_\alpha(x^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha)\otimes_\alpha[1+rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)\right]$$
$$=\mathrm{Re}\left[\left(1+rE_\alpha(ix^\alpha)\right)\otimes_\alpha Ln_\alpha\left(1+rE_\alpha(ix^\alpha)\right)-rE_\alpha(ix^\alpha)\right]$$
$$=[1+rcos_\alpha(x^\alpha)]\otimes_\alpha\frac{1}{2}Ln_\alpha(1+2rcos_\alpha(x^\alpha)+r^2)$$
$$-rsin_\alpha(x^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha)\otimes_\alpha[1+rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)-rcos_\alpha(x^\alpha).$$ (by Lemma 3.1)

And

$$\left({}_0I_x^\alpha\right)\left[rcos_\alpha(x^\alpha)\otimes_\alpha\frac{1}{2}Ln_\alpha(1+2rcos_\alpha(x^\alpha)+r^2)-rsin_\alpha(x^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha)\otimes_\alpha[1+rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)\right]$$
$$=\mathrm{Im}\left[\left(1+rE_\alpha(ix^\alpha)\right)\otimes_\alpha Ln_\alpha\left(1+rE_\alpha(ix^\alpha)\right)-rE_\alpha(ix^\alpha)\right]$$
$$=rsin_\alpha(x^\alpha)\otimes_\alpha\frac{1}{2}Ln_\alpha(1+2rcos_\alpha(x^\alpha)+r^2)$$
$$+[1+rcos_\alpha(x^\alpha)]\otimes_\alpha arctan_\alpha\left(rsin_\alpha(x^\alpha)\otimes_\alpha[1+rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)-rsin_\alpha(x^\alpha).$$ (by Lemma 3.1)

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we use some methods to obtain the closed forms of two types of fractional integrals. Moreover, our results are generalizations of the results in traditional calculus. In the future, we will continue to use our methods to solve the problems in applied mathematics and fractional differential equations.

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